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Remarks about Single Particle Schrödinger Equations

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Abstract

We discuss formal quantization of noumenal and observer canonical formalisms and the relationship between the two.

Our intent in the present note is to discuss briefly the relationships among classical field equations suggested formally as the Schrödinger equations corresponding to some of the canonical formalisms of the previous paper.† We confine ourselves to particles on Minkowski and Newtonian space-times, S_M and S_N.

Formal quantization from a noumenal constraint, such as equation (5.4I), on the extended phase space gives the classical field equation,‡

$$F(x^\mu, p_\mu = -i\partial/\partial x^\mu)\Psi = 0 \tag{1}$$

while from the time-based constraints like equation (5.28I) one gets

$$F_\Omega^{(\pm)}(x^\mu, p_\mu = -i\partial/\partial x^\mu)\psi_\Omega^{(\pm)} = 0 \tag{2}$$

For the free particle on S_M, equation (1) is

$$-(\Box - m^2)\Psi = 0, \qquad m \neq 0 \tag{3}$$

and equation (2) is

$$[-i\partial_0 + (-\nabla^2 + m^2)^{1/2}]\psi_\Omega^{(\pm)} = 0, \qquad m \neq 0 \tag{4}$$

The solutions to equation (4) for times $x^0 > t_1$ are determined from Dirichlet data furnished at $x^0 = t_1$, corresponding to a representation in which the $\psi_\Omega^{(\pm)}(x)$ are supposed to evolve from prescribed sets of initial conditions. This is not true for equation (3) because it is of the second order in the time, a fact which reflects the expression of the noumenal constraint F as the product of two terms linear in p_0.

† R. G. Cawley, pp. 101–122 this volume (hereinafter cited as I). We designate equation references from I in this way: equation ($x.y$I).

‡ We set $c = 1$ and $\hbar = 1$.

Feshbach & Villars (1958) have discussed a physical interpretation of equation (3), basing it on a study of the equation

$$[(\partial - iqA)^2 - m^2]\Psi = 0 \tag{5}$$

where A is a Maxwell field. They interpret the conserved current for equation (5) as a charge current rather than a probability current, after the manner of Pauli & Weisskopf (1934). Equation (5) is also the basis of the analysis of Marx (1969), which produces a probability interpretation. The equation is the result of formal quantization from the constraint,

$$F = (p - qA)^2 + m^2 \approx 0 \tag{6}$$

The interpretation of Feshbach and Villars provides a single particle theory in the absence of coupling, or in a 'perturbative' sense if the coupling is weak. The particle interpretation breaks down in strong fields. The interpretation of Marx is based on use of the Stueckelberg–Feynman Green's function to propagate positive and negative frequency data from pairs of spacelike hyperplanes into the space-time region between. The boundary conditions on the negative frequency amplitude are final state conditions and in general cannot be specified at the time the initial state is prepared. So the resulting theory does not describe the *time* evolution of a set of prepared conditions as a unique process.†

From the present point of view equation (3) is a noumenal wave equation and its physical particle interpretation requires first the separation of the two evolution modes. This is possible because the Klein–Gordon operator decomposes into two commuting factors corresponding to $F(+)$ and $F(-)$, under which Ψ becomes the sum of its positive and negative frequency parts, which we denote by $\Psi^{(\pm)}$. These satisfy

$$F(\pm)\Psi^{(\pm)} = 0 \tag{7}$$

The second step is to turn the wrong mode around by constructing the observer wave equation corresponding to the observer representation of $F(-)$. The result of this procedure is equations (4) with

$$\Psi^{(+)} = \psi_{\Omega}^{(+)}, \qquad \Psi^{(-)} = (\psi_{\Omega}^{(-)})^* \tag{8}$$

with assumptions of suitable boundary conditions. The same procedure cannot be applied to equation (5) because equation (6) is not a noumenal canonical particle constraint, owing to the failure of the factorization condition equation (5.6I).

For the Newtonian examples discussed in I the difficulties associated with equation (5) do not exist. The noumenal wave equation stemming from (1) is

$$[(-i\partial_0 + H_N)^2 + m^2]\Psi = 0 \tag{9}$$

† This may appear at first to make a consistent observation theoretical interpretation of such a scheme impossible. But probably that is not the case if the 'existence' locally of a stable cosmic background of 'retro-particle' amplitude is postulated.

where $H_N = H_N^{(+)}$ as given in equation (11) below. The corresponding pair of equations for the observer representation are

$$(-i\partial_0 + m + H_N^{(\pm)})\psi_\Omega^{(\pm)} = 0 \tag{10}$$

where, according to equation (5.31I),

$$H_N^{(\pm)} = \pm[-(2M)^{-1}(\nabla \mp iq\mathbf{A})^2 + qV], \qquad M \neq 0 \tag{11}$$

The factorization of equation (9) is accomplished in the same way as that of equation (3). Defining $\Psi^{(\pm)}$ as in equation (7), with $F(\pm)$ given through equations (5.1I) and (5.2I), and using† $H_N^{(-)} = -(H_N^{(+)})^*$, we find equation (10) with the same identifications as equation (8).

Substituting

$$\psi = \exp(imx^0)\psi_\Omega^{(+)} \tag{12}$$

into equation (10) (upper sign) gives the Schrödinger equation,

$$(-i\partial_0 + H_N)\psi = 0 \tag{13}$$

The complex conjugate of equation (13) is

$$(i\partial_0 + H_N{}^*)\psi^* = 0 \tag{14}$$

or, using $H_N{}^* = -H_N^{(-)}$,

$$(-i\partial_0 + H_N^{(-)})\psi^* = 0 \tag{15}$$

which is the lower sign of (10) if

$$\psi^* = \exp(imx^0)\psi_\Omega^{(-)} \tag{16}$$

From equations (12) and (16) we see that the two observer wave functions are given by

$$\psi_\Omega^{(+)} = \exp(-imx^0)\psi \qquad \text{and} \qquad \psi_\Omega^{(-)} = \exp(-imx^0)\psi^* \tag{17}$$

According to equations (17) if the time development of $\exp(-imx^0)\psi(x)$ represents a process, then that of $\exp(-imx^0)\psi(x)^*$ describes the corresponding anti-process.

† We use the * to denote complex conjugation.

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